Clifford algebras and universal sets of quantum gates

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In the paper is shown application of Clifford algebras to constructions of universal sets of quantum gates. It is based on well known application of Lie algebras together with especially simple commutation law for Clifford algebras — all elements of basis either commute or anticommute.

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INTRODUCTION

In the paper is discussed algebraic approach to construction of universal set of quantum gates. Quantum gate U for system of n qubits is unitary $2^n \times 2^n$ matrix. It is possible to write $U = e^{iH}$ with H is Hermitian $2^n \times 2^n$ matrix.

Set of quantum gates U_k is *universal* if any unitary matrix can be obtained with given precision as product of the matrices U_k. Algebraic conditions of universality can be described with using Lie algebra $\mathfrak{u}$ of Lie group of unitary matrices [1,2]: if there is set of Hermitian matrices H_k and it is possible to generate basis of space of all Hermitian $2^n \times 2^n$ matrices with using only commutators $i[H, G] \equiv i(HG - GH)$, then $U_k = \exp(i\tau H_k)$ are universal set of quantum gates if τ is small enough.

In the paper is represented new approach to construction of universal set of gates with using both Lie and Clifford algebras. It is possible, because algebra $\mathbb{C}(2^n \times 2^n)$ of all $2^n \times 2^n$ complex matrices is complex Clifford algebra with $2n$ generators, i.e., there are $2n$ matrices Γ_k with property: $\{\Gamma_k, \Gamma_l\} \equiv \Gamma_k\Gamma_l + \Gamma_l\Gamma_k = 2\delta_{kl}\mathbb{1}$, (where $\mathbb{1}$ is unit matrix) and 2^{2n} different *products* of Γ_k generate basis of $\mathbb{C}(2^n \times 2^n)$ [3,4].

The $2n$ matrices Γ_k are not enough to proof of universality, because we may not use arbitrary *products* of Γ_k, but only *commutators*. In the paper is shown, that with using commutators of Γ_k it is possible to generate only $2n^2 + n$ dimensional subspace, but it is enough to add only one element Γ_u and the new set is universal, i.e., it generates full 4^n dimensional space $\mathfrak{u}(2^n)$.

All $2n$ matrices Γ_k may be chosen Hermitian, full complex algebra was used for simplification. The extra one Hermitian matrix is $\Gamma_u = i\Gamma_{123} \equiv i\Gamma_1\Gamma_2\Gamma_3$ or Γ_{1234}, or any such product of three or four different Γ_k.

Constructive proof of universality with using language of Clifford algebras is based on simple commutation law of 4^n basic elements — they either commute or anticommute, because any such element is product of up to $2n$ Γ_k. Direct construction of any $2^n \times 2^n$ matrix $\Gamma_I \equiv \prod_{k \in I} \Gamma_k$ of Clifford basis by commutators of $2n+1$ initial elements is shown below in Sec. I D, *Theorem 1*.

Question about universality is widely investigated [1,2,5–9], but the method discussed in present work has some special property. Constructions of universal set of gates use *only* infinitesimal and continuous symmetries of group $U(2^n)$ and does not need for discrete operations like permutations of qubits or basic vectors related with 'classical limit of quantum circuits'. The properties of discrete, binary transformations of qubits simply emerge here from structure of infinitesimal transformations of Hilbert space, i.e., directly from Hamiltonians, cf. [1,9].

I. CLIFFORD ALGEBRAS

A. General definitions

For n-dimensional vector space with quadratic form (metric) $g(\vec{x})$, Clifford algebra $\mathfrak{A}$ is formal way to represent a square root of $-g(\vec{x})$ [3,4] or, more formally, $-g(\vec{x})\mathbb{1}$ where $\mathbb{1}$ is the unit of algebra $\mathfrak{A}$. The vector space corresponds to n-dimensional subspace $\mathcal{V}$ of $\mathfrak{A}$: $\vec{x} \mapsto \mathbf{x} \equiv \sum_{l=0}^{n-1} x_l \mathfrak{e}_l$, where $\mathbf{x}$, $\mathfrak{e}_l \in \mathcal{V} \subset \mathfrak{A}$. From $\mathbf{x}^2 = -g(\vec{x})$, i.e., $\left(\sum_{l=0}^{n-1} x_l \mathfrak{e}_l\right)^2 = \sum_{i,j=0}^{n-1} g_{ij} x_i x_j$ follows main properties of generators $\mathfrak{e}_l$ of the Clifford algebra:

$$\{\mathfrak{e}_i, \mathfrak{e}_j\} \equiv \mathfrak{e}_i\mathfrak{e}_j + \mathfrak{e}_j\mathfrak{e}_i = -2g_{ij}. \tag{1.1}$$

Let g_{ij} is diagonal and $g_{ii} = \pm 1$ (the case $g_{ii} = 0$ is not considered here, but see [3]). Then:

$$\mathfrak{e}_i\mathfrak{e}_j = -\mathfrak{e}_j\mathfrak{e}_i \quad (i \neq j), \tag{1.2a}$$
$$\mathfrak{e}_i^2 = \pm 1. \tag{1.2b}$$

It is more clear from Eq. (1.2) that it is possible to generate no more than 2^n different products of up to n $\mathfrak{e}_i$. Linear span of all the products is full algebra $\mathfrak{A}$ [4]. Let us use notations $\mathfrak{e}_I = \mathfrak{e}_{i_1 i_2 \ldots i_k} \equiv \mathfrak{e}_{i_1}\mathfrak{e}_{i_2} \cdots \mathfrak{e}_{i_k}$ with k is number of multipliers or *order* of $\mathfrak{e}_I$, $k = \#I$.

If there are no other algebraic relations except of Eq. (1.2), then algebra has maximal dimension 2^n and called *universal* Clifford algebra, $\mathfrak{Cl}(g)$ because for any other Clifford algebra $\mathfrak{A}$ with same metric $g(\vec{x})$ there is homomorphism $\mathfrak{Cl}(g) \to \mathfrak{A}$ (see Ref. [4]).

Let us use notation $\mathfrak{Cl}(l, m)$ for diagonal metric Eq. (1.2) with l pluses and m minuses in Eq. (1.2b), i.e., for pseudo-Euclidean (Minkowski) space $\mathbb{R}^{l,m}$. There is special notation for Euclidean space: $\mathfrak{Cl}(n) \equiv \mathfrak{Cl}(n, 0)$ and $\mathfrak{Cl}_+(n) \equiv \mathfrak{Cl}(0, n)$.

Complexification of any Clifford algebra $\mathfrak{Cl}(l,m)$ with $l+m=n$ is same complex algebra $\mathfrak{Cl}(n,\mathbb{C})$ because all signs in Eq. (1.2b) may be 'adjusted' by substitution $\mathfrak{e}_k \to i\mathfrak{e}_k$.

Let us denote $\mathfrak{e}_I^\sigma \equiv \sqrt{\mathfrak{e}_I^2}\mathfrak{e}_I$, i.e., if $\mathfrak{e}_I^2=1$ then $\mathfrak{e}_I^\sigma=\mathfrak{e}_I$, but if $\mathfrak{e}_I^2=-1$ then $\mathfrak{e}_I^\sigma=i\mathfrak{e}_I$ and so always $(\mathfrak{e}_I^\sigma)^2=1$.

B. Matrix representations

All complex Clifford algebras in even dimension $\mathfrak{Cl}(2n,\mathbb{C})$ are isomorphic with full algebra of $2^n\times 2^n$ complex matrices [3,4]. Simplest case $\mathfrak{Cl}(2,\mathbb{C})$ is Pauli algebra. Matrices σ_x and σ_y can be chosen as generators $\mathfrak{e}_0$, $\mathfrak{e}_1$ and σ_z is $i\mathfrak{e}_0\mathfrak{e}_1=\mathfrak{e}_{01}^\sigma$.

The Pauli algebra is four-dimensional complex algebra and also can be considered as eight-dimensional real algebra, $\mathfrak{Cl}_+(3)$. Prevalent applications of Clifford algebras in theory of NMR quantum computation [10,11] are based rather on real representation $\mathfrak{Cl}_+(3)$ than on complex one $\mathfrak{Cl}(2,\mathbb{C})$ discussed in present work. The two approaches are very close, but may be different in some details.

There is simple recursive construction of complex Clifford algebra with even number of generators $\mathfrak{Cl}(2n,\mathbb{C})$ with using $\mathfrak{Cl}(2,\mathbb{C})$. For $n=1$ it is Pauli algebra and if there is some algebra $\mathfrak{Cl}(2n,\mathbb{C})$ for $n\geq 1$, then

$$\mathfrak{Cl}(2n+2,\mathbb{C}) \cong \mathfrak{Cl}(2n,\mathbb{C}) \otimes \mathfrak{Cl}(2,\mathbb{C}). \tag{1.3}$$

The proof of (1.3): if $\mathfrak{e}_0,\ldots,\mathfrak{e}_{2n-1}$ are $2n$ generators of $\mathfrak{Cl}(2n,\mathbb{C})$ then $\mathbb{1}^{2n}\otimes\mathfrak{e}_0$ and $\mathbb{1}^{2n}\otimes\mathfrak{e}_1$ together with $2n$ elements $\mathfrak{e}_k\otimes\mathfrak{e}_{01}^\sigma$ are $2n+2$ generators of $\mathfrak{Cl}(2n+2,\mathbb{C})$.

The direct construction of $\mathfrak{Cl}(2n,\mathbb{C})$ is [3,4]:

$$\mathit{\Gamma}_{2k} = \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{n-k-1}\otimes\sigma_x\otimes\underbrace{\sigma_z\otimes\cdots\otimes\sigma_z}_{k}, \tag{1.4a}$$

$$\mathit{\Gamma}_{2k+1} = \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{n-k-1}\otimes\sigma_y\otimes\underbrace{\sigma_z\otimes\cdots\otimes\sigma_z}_{k}, \tag{1.4b}$$

with $\mathfrak{e}_l \triangleq \mathit{\Gamma}_l$, $\mathfrak{e}_l^2=\mathbb{1}$, $\forall l\in 0,\ldots,2n-1$. More generally, algebraic properties of elements $\mathfrak{e}_l$ used in the paper are same for different matrix representations $\mathfrak{e}_l \triangleq M\mathit{\Gamma}_l M^{-1}$, where $M\in SU(2^n)$.

C. Spin groups

One of most known physical applications of Clifford algebras is due to *spin groups*. The group has 2-1 isomorphism with orthogonal (or pseudo-orthogonal) group and related with Dirac equation [4] and transformation properties of wave functions in quantum mechanics.

Each element $\mathbf{x}\in\mathcal{V}$ has inverse $\mathbf{x}^{-1}=-\mathbf{x}/g(\vec{x})$ if $g(\vec{x})\neq 0$. All possible products of *even* number of such elements with $|g|=1$ is *spin group*. It is $\mathrm{Spin}(n)$ for $\mathfrak{Cl}(n)$ and for $\mathfrak{Cl}_+(n)$. The group has 2-1 isomorphism with $SO(n)$. For $s\in\mathrm{Spin}(n)$ element of $SO(n)$ is represented as $r_s\colon\mathbf{x}\mapsto s\mathbf{x}s^{-1}$ [4].

Because only products of even number of elements of $\mathfrak{Cl}(n)$ are used in definition of $\mathrm{Spin}(n)$, the group is subset of even subalgebra $\mathfrak{Cl}^e(n)\subset\mathfrak{Cl}(n)$. In Euclidean case $\mathfrak{Cl}^e(n)$ is isomorphic with $\mathfrak{Cl}(n-1)$ and the property let define $\mathrm{Spin}(n+1)$ as subset of $\mathfrak{Cl}(n)$.

The construction of $\mathrm{Spin}(n+1)$ group from $\mathfrak{Cl}(n)$ is sometime called *spoin group* [4] $\mathrm{Spoin}(n)\cong\mathrm{Spin}(n+1)$. It is described here due to application below in the paper.

Let us consider $n+1$ dimensional space $\lambda\mathbb{1}\oplus\mathcal{V}$, i.e., combinations $\mathbf{y}=\lambda+\mathbf{x}$, $\mathbf{x}\in\mathcal{V}$. Let $\Delta(\mathbf{y})\equiv\lambda^2+g(\vec{x})$. The elements have inverse $(\lambda+\mathbf{x})^{-1}=(\lambda-\mathbf{x})/\Delta(\mathbf{y})$ if $\Delta(\mathbf{y})\neq 0$. Products of *any* number of such elements with $|\Delta|=1$ is $\mathrm{Spoin}(n)$ group [4].

The group $\mathrm{Spoin}(n)$ is 2-1 isomorphic with $SO(n+1)$. For $s\in\mathrm{Spoin}(n)$ an element of $SO(n+1)$ is represented as $r_s\colon\mathbf{y}\mapsto s\mathbf{y}(s')^{-1}$, where $\mathbf{y}=y_n+\sum_{l=0}^{n-1}y_l\mathfrak{e}_l$ and $(')$ is the algebra automorphism defined on basis elements as $\mathfrak{e}_I'=(-1)^{\#I}\mathfrak{e}_I$ [4].

D. Lie algebras and Clifford algebras

Clifford algebra is Lie algebra with respect to bracket operation $[a,b]\equiv ab-ba$ [4]. Here is proved some result, necessary for construction of universal set of gates.

Theorem 1. Let $\mathfrak{Cl}(n,\mathbb{C})$ is Clifford algebra and n is even. It is enough n generators $\mathfrak{e}_k$, $k=0,\ldots,n-1$ and any element $\mathfrak{e}_I$ with $\#I=3$ or $\#I=4$ to generate element of any order only with using commutators of the $n+1$ elements.

The proof of the result has few steps.

(1) If there are n elements $\mathfrak{e}_0,\ldots,\mathfrak{e}_{n-1}$, it is possible with using commutators to generate also all elements of second order, i.e., $[\mathfrak{e}_i,\mathfrak{e}_j]=2\mathfrak{e}_i\mathfrak{e}_j\equiv 2\mathfrak{e}_{ij}$.

(2) If we have an element of third order, for example $\mathfrak{e}_{012}$, it is possible to generate any element of third order. For example $2\mathfrak{e}_{01m}=[\mathfrak{e}_{012},\mathfrak{e}_{2m}]$, $2\mathfrak{e}_{0nm}=[\mathfrak{e}_{01m},\mathfrak{e}_{1n}]$, $2\mathfrak{e}_{pnm}=[\mathfrak{e}_{0nm},\mathfrak{e}_{0p}]$.

(3) Analogously, if there is any element of order $2k+1$ it is possible to generate any element of same order with using no more than $2k+1$ commutators with elements $\mathfrak{e}_{ij}$.

(4) If we have all elements of third order, it is possible to generate any element of fourth order, $2\mathfrak{e}_{ijkl}=[\mathfrak{e}_{ijk},\mathfrak{e}_l]$.

(5) Analogously, if we have all elements with order $\#I=2k+1$, it is possible to generate any element of order $2k+2$, $2\mathfrak{e}_{I\cup l}=[\mathfrak{e}_I,\mathfrak{e}_l]$, where $l\notin I$.

(6) If we have any element of fourth order it is possible to generate some element of third order, $2\mathfrak{e}_{ijk}=[\mathfrak{e}_{ijkl},\mathfrak{e}_l]$ (and so any element of third and fourth order).

(7) Analogously, if we have any element of order $2k+2$ it is possible to generate some element of order $2k+1$ (and so any element with order $2k+1$ and $2k+2$ as in steps 3 and 5).

(8) We have all elements with order less or equal to $2k$, $k\geq 2$ due to steps 1,2,4 and we can prove theorem by recursion: by using commutator of element with order

$2k-1$ and element with order 3 it is possible to generate an element of order $2k+2$ and so any elements of order $2k+1$ and $2k+2$ as in step 7. $\square$

Note 1. Instead of elements $\mathfrak{e}_0, \ldots, \mathfrak{e}_{n-1}$ it is possible to use $\mathfrak{e}_0$ together with $n-1$ elements $\mathfrak{e}_{l-1,l}$: $[\mathfrak{e}_0, \mathfrak{e}_{01}] = 2\mathfrak{e}_1, \ldots, [\mathfrak{e}_{l-1}, \mathfrak{e}_{l-1,l}] = 2\mathfrak{e}_l$.

Note 2. If n is odd, it is impossible to generate only element with order n, because due to step 7 of recursion it would be generated only from even element with order $n+1$, but there is no such elements. So in this case we need $n+2$ elements, extra one is: $\mathfrak{e}_{0,\ldots,n-1}$.

Note 3. If to use only n generators $\mathfrak{e}_i$, then together with $n(n-1)/2$ commutators $[\mathfrak{e}_k, \mathfrak{e}_j] = 2\mathfrak{e}_{kj}$, $k \neq j$, it is possible to generate $n+n(n-1)/2 = n(n+1)/2$ elements, because it may be checked directly, any new commutators may not generate an element with order more than two. It is Lie algebra for $\mathrm{Spoin}(n)$ group, because products of $\exp(\epsilon\mathfrak{e}_k) \approx \mathbb{1} + \epsilon\mathfrak{e}_k$ belong to the group and dimension of the group is the same $\dim \mathrm{Spoin}(n) = \dim SO(n+1) = n(n+1)/2$. Despite of only elements $\mathfrak{e}_I$, $\#I \le 2$ belong to the *Lie algebra*, all 4^n elements $\mathfrak{e}_I$, $\#I \le n$ of $\mathfrak{Cl}(n)$ belong to *Lie group* $\mathrm{Spoin}(n)$ by definition and so linear span of the elements is full Clifford algebra.

Note 4. The theorem was proved rather for more general case of Lie algebra of complex Lie group $GL(N,\mathbb{C})$, $N = 2^{n/2}$ of all matrices $M, \det(M) \neq 0$, than for unitary group $U(N) \subset GL(N,\mathbb{C})$. The proof for Lie algebra $\mathfrak{u}(N)$ of unitary group $U(N)$ is direct implication, it is enough to choose initial matrices in $\mathfrak{u}(N)$ for given representation and then Lie brackets may produce only matrices in $\mathfrak{u}(N)$ for each step of proof.

It should be mentioned, there are two traditions for representations of $\mathfrak{u}(N)$. In physical applications are used Hermitian matrices H, Lie brackets is $i[a,b]$, and unitary matrices are represented as $U = \exp(-i\tau H)$ due to relations with Hamiltonians and quantum version of Poisson brackets [12]. In representation Eq. (1.4) elements $\mathfrak{e}_l = \mathit{\Gamma}_l$, $i\mathfrak{e}_{012}$ and $\mathfrak{e}_{0123}$ (and $i\mathfrak{e}_{kl}$, see *Note 1*), i.e., all $\mathfrak{e}_I^\sigma$ are Hermitian. In more general mathematical applications $\mathfrak{u}(N)$ are skew-Hermitian matrices $A^\dagger = -A$ and 'i' multipliers are not present in expressions for commutators and exponents [4], because $A \triangleq iH$.

II. APPLICATION TO QUANTUM GATES

A. Universal set of quantum gates

Now let us discuss construction of universal gates more directly. Instead of Lie algebra $\mathfrak{u}(2^n)$ we should work with Lie group $U(2^n)$ and then element $\mathfrak{e}_I^\sigma$ corresponds to unitary gate $U_I^\tau \equiv \exp(i\tau\mathfrak{e}_I^\sigma)$. One of advantages of elements $\mathfrak{e}_I^\sigma$ is analytical expression for the exponent:

$$U_I^\tau = e^{i\tau\mathfrak{e}_I^\sigma} = \cos(\tau) + i\sin(\tau)\mathfrak{e}_I^\sigma. \tag{2.1}$$

The Eq. (2.1) is valid for any operator with property $\mathfrak{e}^2 = \mathbb{1}$ and it is true for all 4^n basis elements $\mathfrak{e}_I^\sigma$.

It is also possible due to Eq. (2.1) to combine approach with *infinitesimal* parameters τ [1,2] and approach with *irrational* parameter [5,6]. The less τ, the higher precision in generation of arbitrary unitary gates in [1,2]. Due to Eq. (2.1) the accuracy may be arbitrary high if we use gates $U_I = e^{i\varpi\mathfrak{e}_I^\sigma}$ with irrational ϖ/π because for any τ there exist natural number N and $\varepsilon < \tau$: $U_I^\varepsilon = (U_I)^N$. It should be mentioned, the unitary gates do not necessary have irrational coefficients even if ϖ/π is irrational, for example let $U_I = 0.8 + 0.6\mathfrak{e}_I^\sigma$.

Yet another advantage of elements $\mathfrak{e}_I^\sigma$ is more simple expression for 'commutator gate'. In usual case [1,2] it is generated as

$$e^{i\tau i[H_k,H_l]} \approx e^{i\sqrt{\tau}H_k}e^{-i\sqrt{\tau}H_l}e^{i\sqrt{\tau}H_k}e^{-i\sqrt{\tau}H_l},$$

and the expression has precision $O(\tau^{1.5})$. For elements $\mathfrak{e}_I^\sigma$ there is exact construction, if $H_I = \mathfrak{e}_I^\sigma$ and $H_J = \mathfrak{e}_J^\sigma$ then either $[H_I, H_J] = 0$ or $[H_I, H_J] = 2H_IH_J$. First case is trivial and for second case due to Eq. (2.1):

$$e^{i\tau i[H_I,H_J]/2} = e^{-\tau H_I H_J} = e^{i\frac{\pi}{2}H_I}e^{i\tau H_J}e^{-i\frac{\pi}{2}H_I}.$$

After construction of basis of Hermitian matrices $H_I = \mathfrak{e}_I^\sigma$ it is possible to use expression:

$$e^{\sum_I \alpha_I H_I} = \left(e^{\frac{1}{N}\sum_I \alpha_I H_I}\right)^N \approx \left(\prod e^{\frac{1}{N}\alpha_I H_I}\right)^N \equiv \left(\prod U_I^{\frac{\alpha_I}{N}}\right)^N.$$

The expression has accuracy $O\left(\sum \alpha_I^2/N\right)$.

The approach to universal set of gates U is more convenient and constructive if we know Hermitian matrix H, $U^\tau = e^{i\tau H}$. It is not principle limitation, because for physical realizations we should know Hamiltonian to construct the gates. It is also related with universal quantum simulation [9] with H is Hamiltonian and τ is real continuous parameter — time of 'application'.

The description with exponent maybe even more complete, because with using H it is possible to find unique $U = \exp(iH)$, but with using U it is not always possible to restore H because there are many H for same U. A simple example is $U = i\sigma_\alpha \otimes \sigma_\beta$ with arbitrary two Pauli matrices: $U = e^{i\pi\,(\sigma_\alpha\otimes 1 + 1\otimes\sigma_\beta)} = e^{i\pi\,\sigma_\alpha\otimes\sigma_\beta}$.

B. Two-qubit quantum gates.

Let us show how to build universal set of one- and two-qubit gates with using representation Eq. (1.4). For example it may be $2n+1$ gates $\exp(i\tau\mathfrak{e}_I)$, where $\mathfrak{e}_I$ are $\mathfrak{e}_0$, $i\mathfrak{e}_{l-1,l}$ with $l = 1, \ldots, 2n-1$, and $i\mathfrak{e}_{012}$:

$$\mathfrak{e}_0 = \mathbb{1}^{\otimes(n-1)} \otimes \sigma_x, \tag{2.2a}$$
$$\tfrac{1}{i}\mathfrak{e}_{2k,2k+1} = \mathbb{1}^{\otimes(n-k-1)} \otimes \sigma_z \otimes \mathbb{1}^{\otimes k}, \tag{2.2b}$$
$$\tfrac{1}{i}\mathfrak{e}_{2k+1,2k+2} = \mathbb{1}^{\otimes(n-k-2)} \otimes \sigma_x \otimes \sigma_x \otimes \mathbb{1}^{\otimes k}, \tag{2.2c}$$
$$\tfrac{1}{i}\mathfrak{e}_{012} = \mathbb{1}^{\otimes(n-2)} \otimes \sigma_x \otimes \mathbb{1} \tag{2.2d}$$

with $k = 0, \ldots, n-1$ or $n-2$. The elements was discussed in *Note 1* and it was shown, they generate full Lie algebra $\mathfrak{u}(2^n)$.

C. Nonuniversal set of quantum gates

In the [2] has arisen an interesting question, which set of gates are *not* universal (and why).

Products of gates $U_k^\tau = e^{i\tau\mathfrak{e}_k} = \cos(\tau) + i\mathfrak{e}_k \sin(\tau)$ generate group $\mathrm{Spin}(2n+1) \cong \mathrm{Spoin}(2n) \subset U(2^n)$ due to *Note 3*. It is interesting example of nonuniversality, then only one extra gate like $e^{i\tau\mathfrak{e}^\sigma_{012}}$ may produce universal set with 'an exponential improvement' from subgroup $\dim \mathrm{Spoin}(2n) = n(2n+1)$ to full group $\dim U(2^n) = 2^{2n}$.

This result is more important, if the extra gate $e^{i\tau\mathfrak{e}^\sigma_I}$ with $\#I = 3$ or $\#I = 4$ has other physical nature, than gates with $\#I = 1$ and $\#I = 2$. It is not clear from expression Eq. (2.2) there the extra gate Eq. (2.2d) is simply 1-gate. But it is not so for physical systems with *natural* Clifford and spin structure.

A possible reason is Schrödinger equation for n particles without interaction [13]: $i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2}\hbar^2 \sum_{a=1}^n \frac{\Delta_a}{m_a}\psi$, or with using $m_a = m$, Laplacian Δ_N with $N = \nu n$ variables it is possible to write for stationary solution with total energy E :

$$(\Delta_N + \lambda^2)\psi(x_0, \ldots, x_{N-1}) = 0, \qquad (2.3)$$

where $\lambda \equiv \sqrt{2mE}/\hbar$. Let dimension of one particle motion $\nu = 2$ for simplicity, $N = 2n$.

Let us consider full basis $\phi_{\mathbf{p}}(\mathbf{x}) \equiv e^{i(\mathbf{p},\mathbf{x})}$ on Hilbert space $\mathcal{L}$ of wave functions $\psi \in \mathcal{L}$. Here $\mathbf{p}, \mathbf{x} \in \mathbb{R}^N$ and $(\mathbf{p}, \mathbf{x})$ is scalar product. The plane waves $\phi_{\mathbf{p}}$ correspond to n particles with definite momentum. If $O \in SO(N)$, then transformation defined on basis as $\Sigma_O\colon \phi_{\mathbf{p}} \to \phi_{O\mathbf{p}}$ is *symmetry* of Eq. (2.3). It is analogue of classical transition between two configurations *with same total kinetic energy* in "billiard balls" conservative logic [14].

General Dirac operator [4] is first order differential operator $\mathfrak{D}_N = \sum_{i=0}^{N-1} i\mathfrak{e}_k \frac{\partial}{\partial x_k}$ with property $\mathfrak{D}_N^2 = -\Delta_N$. If to use Dirac operator for factorisation of Eq. (2.3)

$$(\mathfrak{D}_N - \lambda)(\mathfrak{D}_N + \lambda)\mathbf{\Psi}(x_0, \ldots, x_{N-1}) = 0, \qquad (2.4)$$

then each component of $\mathbf{\Psi}$ is solution of Eq. (2.3) and action of $\mathrm{Spin}(N)$ group on $\mathbf{\Psi}$ corresponds [4] to $SO(N)$ symmetry Σ_O described above and it has some analogue in classical physics of billiard balls. The $\mathrm{Spoin}(N)$ group is represented less directly, but can be considered as symmetry between two stationary solutions with *different* total energies.

The example above shows, it is possible to find some classical correspondence for elements $\mathfrak{e}_I$, $\#I = 2$ of Spin group and maybe for generators $\#I = 1$ of Spoin group, but special element with $\#I = 3$ does not have some allusion with classical physics.

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